Вселенная Гёделя как группа Ли с левоинвариантной лоренцевой метрикой и разложение Ивасавы

В. Н. Берестовский

Институт математики имени С.Л.Соболева СО РАН,
Новосибирский государственный университет

Международная научная конференция
"Комбинаторно-вычислительные методы алгебры и логики",
посвященная памяти Виталия Анатольевича Романькова

Омск, 15–19 июля, 2024

Список литературы

1. *Gödel K.* An example of a new type of cosmological solutions of Einstein's field equations of gravitation. Rev. Mod. Phys., 21:3 (1949), 447–450.

2. *Hilgert Jo., Neeb K.–H.* Lie semigroups and their Applications. Lect. Notes Math., 1552. Springer-Verlag: Berlin, Heidelberg, 1993.

3. *Agrachev A., Barilari D.* Sub-Riemnnian structures on 3d Lie groups. Journal of Dynamical and Control Systems, 18:1 (2012), 21–44. DOI:10.1007/s10883-012-9133-8

Список литературы

4. *Falbel E., Gorodski C.* Sub-Riemannian homogeneous spaces in dimensions 3 and 4. Geom.Dedicata 62:3(1996), 227–252.

5. *Helgason S.* Differential Geometry, Lie groups and Symmetric Spaces. Graduate Studies in Mathematics, Vol. 34. Providence, R.I.: AMS, 2001.

6. *Goto M., Grosshans F.D.* Semisimple Lie algebras. Lecture Notes in Pure and Applied Mathematics, Vol. 38. Marcel Dekker, Inc. New York and Basel, 1978.

7. *Gromoll D., Klingenberg W., Meyer W.* Riemannsche Geometrie im Grossen. Lect. Notes Math., 55. Springer-Verlag, Heidelberg, 1968.

Abstract

We discuss a model of the Gödel Universe as Lie groups with left-invariant Lorentz metric for simply connected four-dimensional Lie group, the Iwasawa decomposition for semisimple Lie groups, and left-invariant Lorentz metrics on $SL(2,\mathbb{R})$ and $\widetilde{SL(2,\mathbb{R})}$, following K.-H. Neeb. Also we show that the isometry between two non-isomorphic sub-Riemannian Lie group, constructed by A. Agrachev and D. Barilari, is induced by some Iwasawa decomposition of $SL(2,\mathbb{R})$.

Introduction

Kurt Gödel in paper [1] of 1949 introduced the Lorentz metric of the signature $(+,-,-,-)$ on the space $\mathbb{R}^4$. The Gödel Universe (space-time) S is a solution of the General Relativity Theory (the Einstein gravitation equations).

Professor Karl–Hermann Neeb wrote to the author that it is possible to realize the Gödel Universe otherwise and sent an electronic version of his joint with Joachim Hilgert book [2], where in section 2.7 ''Gödel's cosmological model and universal covering of $SL(2,\mathbb{R})$'' is suggested a left-invariant Lorentz metric on this Lie group.

In this connection, it is useful to mention the paper [3] by A. Agrachev and D. Barilari, where the autors obtained a full classification of left-invariant sub-Riemannian metrics on three-dimensional Lie groups and "explicitly find a sub-Riemannian isometry between nonisomorphic Lie groups $SL(2,\mathbb{R})$ and $SO(2)\times A^{+}(\mathbb{R})$" [3].

The existence of such isometry was indicated ealier in [4] by Falbel and Gorodski.

In a message to the author, Professor Neeb explains this by a diffeomormism of Lie groups $SL(2,\mathbb{R})$ and $SO(2)\times A^{+}(\mathbb{R})$ by means of the Iwasawa decomposition for $SL(2,\mathbb{R})$. Let us cite now only Theorems 6.5.1 and 9.1.3 from [5].

In p. 10.6.4 (i) from [5] are indicated the isomorphisms of Lie algebras:

$$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1) \cong \mathfrak{sp}(1,\mathbb{R}).$$

Then simply connected Lie groups with these Lie algebras are isomorphic.

In Theorem 3 we prove some properties of special left-invariant Lorentz metrics on three-dimensional Lie groups. Also we show in Proposition 3 that the isometry between two non-isomorphic sub-Riemannian Lie group, constructed by A. Agrachev and D. Barilari, is induced by some Iwasawa decomposition of $SL(2,\mathbb{R})$.

The Goedel Universe as a Lie group with left-invariant Lorentz metric

Gödel introduced in [1] his space-time S as $\mathbb{R}^4$ with the linear element

$$ds^2 = a^2 \left(dx_0^2 + 2e^{x_1} dx_0 dx_2 + \frac{e^{2x_1}}{2} dx_2^2 - dx_1^2 - dx_3^2 \right), \quad a > 0. \quad (1)$$

Gödel noticed that it is possible to rewrite this quadratic form in view of

$$ds^2 = a^2 \left[(dx_0 + e^{x_1} dx_2)^2 - dx_1^2 - \frac{e^{2x_1}}{2} dx_2^2 - dx_3^2 \right], \quad (2)$$

which shows obvious that its signature is equal everywhere to $(+,-,-,-)$.

We shall assume that $a = 1$.

Gödel noticed in [1] that on (S, ds^2) acts simply transitively a four-dimensional isometry Lie group. It is easy to see that such action could be written as

$$y_0 = x_0' + x_0, \quad y_1 = x_1' + x_1, \quad y_2 = x_2' e^{-x_1} + x_2, \quad y_3 = x_3' + x_3 \quad (3)$$

with arbitrary $x_0, x_1, x_2, x_3 \in \mathbb{R}$. This implies that corresponding Lie group G is the simplest simply connected noncommutative four-dimensional Lie group of the view

$$G \cong [(\mathbb{R}, +) \times G_2] \times (\mathbb{R}, +) := G_3 \times (\mathbb{R}, +), \qquad (4)$$

where G_2 is unique up to isomorphism, necessary isomorphic to $\mathbb{R}^2$, two-dimensional noncommutative Lie group. The Lie group G_2 (G_3) is isomorphic to the Lie group $A^+(\mathbb{R})$ ($(\mathbb{R}, +) \times A^+(\mathbb{R})$), where $A^+(\mathbb{R})$ is the group of preserving orientation affine transformations of $(\mathbb{R}, +)$.

In case under consideration, identifying the quad (x_0', x_1', x_2', x_3') with the vector $(x_2', x_1', x_0', x_3', 1)^T$, where ^{T} is the sign of transposition, the action of the group G on $\mathbb{R}^4$ by formula (3) has the view $(y_2, y_1, y_0, y_3, 1)^T = A(x_2', x_1', x_0', x_3', 1)^T$, where

$$A = \begin{pmatrix} e^{-x_1} & 0 & 0 & 0 & x_2 \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & 0 & x_0 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Under this the equality

$$A(0,0,0,0,1)^T = (x_2, x_1, x_0, x_3, 1)^T \tag{6}$$

sets the bijection of the group G onto $\mathbb{R}^4$ and the unit of G corresponds to the zero-vector $(0,0,0,0) \in \mathbb{R}^4$.

On base of this, (4) and (1), we can identify (S, ds^2) with the Lie group G equipped with left-invariant Lorentz metric. Let

$$e_0 = \frac{\partial}{\partial x_0}(0), e_1 = \frac{\partial}{\partial x_1}(0), e_2 = \frac{\partial}{\partial x_2}(0), e_3 = \frac{\partial}{\partial x_3}(0))$$

be the basis of the Lie algebra $\mathfrak{g}$ of the Lie group G at the unit of G, corresponding to coordinates (x_0, x_1, x_2, x_3). Then, according to what has been said and (1), all nonzero components of the linear element ds^2 with respect to this basis are equal to

$$g_{00} = 1, g_{02} = g_{20} = 1, g_{22} = \frac{1}{2}, g_{11} = -1, g_{33} = -1. \qquad (7)$$

According to (6), the Lie subgroup G_3 can be identified with the matrix Lie group

$$\begin{pmatrix} e^{-x_1} & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (x_0, x_1, x_2) \in \mathbb{R}^3. \qquad (8)$$

It is obvious that $(S, ds^2) = (S_0, ds_0^2) \times (S_1, ds_1^2)$, where $S_0 = \mathbb{R}^3$, $S_1 = \mathbb{R}$,

$$ds_0^2 = dx_0^2 + 2e^{x_1}dx_0dx_2 + \frac{e^{2x_1}}{2}dx_2^2 - dx_1^2, \quad ds_1^2 = -dx_3^2. \tag{9}$$

Also it is clear that we can consider (S_0, ds_0^2) as the matrix Lie group (8) with left-invariant Lorentz metric, which according to (7) has nonzero components

$$g_{00} = 1, g_{02} = g_{20} = 1, g_{22} = \frac{1}{2}, g_{11} = -1 \tag{10}$$

with respect to the basis e_0, e_1, e_2 of the Lie algebra $\mathfrak{g}_3$ of the matrix Lie group (8).

In consequence of (6), for the Lie algebra $\mathfrak{g}_3$ of the Lie group G_3,

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

Then in the Lie algebra $\mathfrak{g}_3$,

$$[e_1, e_2] = e_1 e_2 - e_2 e_1 = -e_2, \quad [e_0, e_1] = [e_0, e_2] = 0. \tag{12}$$

The Iwasawa decomposition of Lie algebras and Lie groups

Let $\mathfrak{g}$ be a semisimple real Lie algebra, σ be some Cartan involution of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition ($\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$, consisting of fixed points relative to σ). Let us denote by $mathfraka$ a maximal commutative subspace in $\mathfrak{p}$. Then there is the following *Iwasawa decomposition of Lie algebra* $\mathfrak{g}$.

Theorem 1

(4.7.2) in [6]. Let $\mathfrak{g}$ be a semisimple real Lie algebra.
Then there exists a direct sum of vector subspaces in $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad (13)$$

where $\mathfrak{n}$ is a nilpotent subalgebra in $\mathfrak{g}$ such that the endomorphism adX is nilpotent for every $X \in \mathfrak{n}$, and $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra in $\mathfrak{g}$.

As an example, the authors of [6] give the decomposition (13) for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. In this case $\mathfrak{k}$ is the Lie subalgebra of skew-symmetric matrices, $\mathfrak{a}$ is the Lie subalgebra of diagonal matrices (with zero trace), and $\mathfrak{n}$ is the Lie subalgebra of strictly upper triangular matrices. In particular, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \right\}, \quad \mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \right\}, \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right\}, \quad t \in \mathbb{R}, \tag{14}$$

with natural basis

$$f_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{15}$$

and Lie brackets for this basis

$$[f_0, f_1] = 2f_0 - 3f_2, \quad [f_0, f_2] = f_1, \quad [f_1, f_2] = 2f_2. \tag{16}$$

Let $K = \exp(\mathfrak{k})$, $A = \exp(\mathfrak{a})$, $N = \exp(\mathfrak{n})$ be Lie subgroups of the semisimple Lie group G, corresponding to the decomposition (13).

Theorem 2

(Theorem 9.1.3 in [5]) Let G be a connected semisimple real Lie group. Then $G = KAN$ and the mapping

$$(k, a, n) \to kan \tag{17}$$

is the diffeomorphism of manifold $K \times A \times N$ onto the Lie group G.

Corollary 1

The Lie group G is diffeomorphic to Lie groups $K \times AN$ and $K \times A \times N$.

Theorem 1 and Theorem 6.1.1 from [5] imply the following

Proposition 1

The sets K, A, N, and AN are connected closed Lie subgroups of the Lie group G, where $Ad_G(K)$ is compact, A is commutative, N is nilpotent, and AN is solvable. The subgroup K contains the center Z of the Lie group G. In addition, K is compact if and only if the center Z of G is finite; in this case K is a maximal compact subgroup of the Lie group G.

Corollary 2

If $G = SL(n, \mathbb{R})$, then $K = SO(n)$, A is the group of all real diagonal $(n \times n)$-matrices with unit determinant, N could be considered as the group of all real upper triangular $(n \times n)$-matrices with units on the main diagonal, and $Sol(n) := AN$ as the group of all real upper triangular $(n \times n)$-matrices with unit determinant.

Corollary 3

The Lie group $SL(n, \mathbb{R})$ is diffeomorphic to Lie groups $SO(n) \times Sol(n)$ and $SO(n) \times A \times N$. So $SL(2, \mathbb{R})$ is diffeomorphic to $SO(2) \times Sol(2)$ and to commutative Lie group $SO(2) \times A \times N$.

Isomorphism of Lie groups G_3 and $(\mathbb{R},+)\times A^+(\mathbb{R})$

Proposition 2

There exist an isomorphism of the Lie group $G_3=(\mathbb{R},+)\times A^+(\mathbb{R})$ onto the Lie group $(\mathbb{R},+)\times Sol(2)$ and corresponding realization of (S_0, ds_0^2) as the Lie group $(\mathbb{R},+)\times Sol(2)$ with left-invariant Lorentz metric.

Proof. Comparing (12) and (16), we see that the linear map $\varphi:\mathfrak{a}\oplus\mathfrak{n}\to\mathfrak{g}$ such that

$$\varphi\left(\frac{-f_1}{2}\right)=e_1,\quad \varphi(f_2)=e_2 \tag{18}$$

is an isomorphism of Lie algebras. Let $Sol(2)$ be the Lie group with the Lie algebra $\mathfrak{a}\oplus\mathfrak{n}$ for (14). Then (18) defines isomorphism of Lie groups $\psi: Sol(2)\to G_2$:

$$\psi\left(\left(\begin{array}{cc} e^{-s/2} & 0 \\ 0 & e^{s/2} \end{array}\right)\left(\begin{array}{cc} 1 & r \\ 0 & 1 \end{array}\right)\right)=\left(\begin{array}{cccc} e^{-s} & 0 & 0 & e^{-s}r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad (19)$$

$$\psi\left(\left(\begin{array}{cc} 1 & r \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} e^{-s/2} & 0 \\ 0 & e^{s/2} \end{array}\right)\right)=\left(\begin{array}{cccc} e^{-s} & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \quad (20)$$

We can consider $(\mathbb{R},+)$ as Lie algebra and as Lie group. Then mappings

$$t\in(\mathbb{R},+)\to\left(\begin{array}{cc} 0 & t \\ -t & 0 \end{array}\right), \quad t\in(\mathbb{R},+)\to\left(\begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array}\right) \quad (21)$$

are correspondingly the isomorphism of Lie algebras and respective universal covering epimorpism of Lie groups.

Then there exists unique isomorphism ψ of the Lie group $(\mathbb{R}, +) \times Sol(2)$ onto the Lie group G_3, with properties (19), (20), and $\psi(t) = t$ for $t \in (\mathbb{R}, +)$. It follows from previous considerations that we shall realize (S_0, ds_0^2) as the Lie group $(\mathbb{R}, +) \times Sol(2)$ with left-invariant Lorentz metric if components of this metric in the basis $\{f_0, -f_1/2, f_2\}$ of its Lie algebra will be as in (10).
The corresponding orthonormal basis is

$$X = f_0, \quad Y = -f_1/2, \quad Z = \sqrt{2}(f_0 - f_2). \tag{22}$$

Left-invariant Lorentz metrics on $SO(2) \times Sol(2)$ and $SL(2,\mathbb{R})$

Theorem 3

The Lorentz metric on $\widetilde{SL(2,\mathbb{R})}$ from [2] is not isometric to the subspace (S_0, ds_0^2) of the Gödel Universe and the Iwasawa diffeomorphism of $SL(2,\mathbb{R})$ onto $SO(2) \times Sol(2)$ is not isometry for Lorentz metric on $SL(2,\mathbb{R})$ from [2].

Proof.
For any (pseudo)-Riemannian manifold M with (pseudo)-metric tensor $(\cdot,\cdot)$, the Levi-Civita connection ∇, and smooth vector fields X, Y, Z it follows from equation (3.5.(7)) in [7] that $(\nabla_X Y, Z)$ is equal to

$$\frac{1}{2}[X(Y,Z)+Y(Z,X)-Z(X,Y)+(Z,[X,Y])+(Y,[Z,X])-(X,[Y,Z])]. \tag{23}$$

As a consequence, if $(M, (\cdot, \cdot))$ is a Lie group G with left-invariant (pseudo)-metric $(\cdot, \cdot)$ and X, Y, Z are left-invariant, then

$$(\nabla_X Y, Z) = \frac{1}{2}[(Z, [X, Y]) + (Y, [Z, X]) - (X, [Y, Z])]. \qquad (24)$$

Let assume now that $G = SO(2) \times Sol(2)$ with left-invariant Lorentz metric and (X, Y, Z) be an orthonormal basis of left-invariant vector fields such that X be tangent to $SO(2)$ and $(X, X) = 1$. Then $Y = \alpha X + Y_1$, $Z = \beta X + Z_1$, where $Y_1, Z_1 \in \mathfrak{sol}(2)$ and

$$(\nabla_X Y, Z) = \frac{-1}{2}(X, [Y, Z]) = \frac{-1}{2}(X, W_1) := -\gamma_1, \quad W_1 \in \mathfrak{n}, \qquad (25)$$

$$(\nabla_X Y, X) = 0, \quad (\nabla_X Y, Y) = 0, \quad \nabla_X Y = \gamma_1 Z.$$

We want to calculate

$$R(X, Y)Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y.$$

It follows from (24) that

$$(\nabla_Y Y, X) = (\nabla_Y Y, Y) = 0, \quad (\nabla_Y Y, Z) = -(Y, W_1) := -\gamma_2, \quad \nabla_Y Y = \gamma_2 Z,$$

$$(\nabla_X Z, X) = (\nabla_X Z, Z) = 0, (\nabla_X Z, Y) = \frac{-1}{2}(X, [Z, Y]) = \gamma_1, \nabla_X Z = -\gamma_1 Y,$$

$$(\nabla_Y Z, X) = \gamma_1, \quad (\nabla_Y Z, Y) = \gamma_2, \quad (\nabla_Y Z, Z) = 0, \quad \nabla_Y Z = \gamma_1 X - \gamma_2 Y,$$

$$R(X, Y)Y = \nabla_X(\gamma_2 Z) - \nabla_Y(\gamma_1 Z) = -\gamma_1\gamma_2 Y - \gamma_1^2 X + \gamma_1\gamma_2 Y = -\gamma_1^2 X.$$

Let us calculate yet an analogue of the sectional curvature

$$(R(X, Y)Y, X) = -\gamma_1^2.$$

In [2], the authors consider the Lie group $SL(2,\mathbb{R})$ with left-invariant Lorentz metric and orthonormal basis with opposite signature of the form

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}). \tag{26}$$

Then $[Y, Z] = 2\sqrt{2}X$ in $\mathfrak{sl}(2,\mathbb{R})$, but $[Y, Z] = 0$ in the Lie algebra $\mathfrak{so}(2) \oplus \mathfrak{sol}(2)$, and according to the above, $\gamma_1 = 0$ in $SO(2) \times Sol(2)$ and

$$R(X, Y)Y = 0. \tag{27}$$

Now let us calculate $R(X,Y)Y$ and $(R(X,Y)Y,X)$ for left-invariant Lorentz metric on the Lie group $SO(2)\times Sol(2)$, locally isometric to the the three-dimensional subspace (S_0,ds_0^2) of the Gödel Universe. According to (33) and (10), we have

$$W_1=[Y,Z]=\sqrt{2}[-f_1/2,(f_0-f_2)]=-\sqrt{2}f_2,\quad \gamma_1=-\frac{1}{2}(X,W_1)=\sqrt{2}.$$

So, according to previous calculations,

$$R(X,Y)Y=-2X,\quad (R(X,Y)Y,X)=-2. \tag{28}$$

Let us compute now $R(X,Y)Y$ for left-invariant Lorentz metric on $SL(2,\mathbb{R})$, defined in [2]. It follows from (26) that

$$[X,Y]=\sqrt{2}Z, \quad [X,Z]=\sqrt{2}Y, \quad [Y,Z]=2\sqrt{2}X. \tag{29}$$

Let us apply (24) and (29) in the further computations.

$$(\nabla_X Y, Z) = -\sqrt{2}, \quad (\nabla_X Y, X) = (\nabla_X Y, Y) = 0, \quad \nabla_X Y = \sqrt{2}Z,$$

$$(\nabla_Y Y, X) = (\nabla_Y Y, Y) = (\nabla_Y Y, Z) = 0, \quad \nabla_Y Y = 0,$$

$$(\nabla_Z Y, X) = -2\sqrt{2}, \quad (\nabla_Z Y, Y) = (\nabla_Z Y, Z) = 0, \quad \nabla_Z Y = -2\sqrt{2}X,$$

$$\nabla_Y Z = \nabla_Z Y + [Y,Z] = 0,$$

$$R(X,Y)Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y = -\sqrt{2}(\nabla_Y Z - \nabla_Z Y) = -4X \tag{30}$$

All equalities (27), (28), and (30) are pairwise different.
This implies all statements of Theorem 3.

Isometry of non-isomorphic sub-Riemannian Lie groups

Let us change notation $x_1 \leftrightarrow x_2$, $x_0 \to x_3$. Then the mapping

$$\begin{pmatrix} e^{-x_2} & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} e^{-x_2} & 0 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in A^+(\mathbb{R}) \times (\mathbb{R}, +) \quad (31)$$

is an isomorphism of matrix Lie groups with the basis of the Lie algebra

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

such that only $[e_1, e_2] = -[e_2, e_1] = e_1$ are unique nonzero Lie brackets.

Analogously to (19) and (20), the mapping

$$F\left(\left(\begin{array}{ccc} e^{-x_2} & 0 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{array}\right)\right) = \left(\left(\begin{array}{cc} e^{-x_2/2} & x_1 \\ 0 & e^{x_2/2} \end{array}\right), \left(\begin{array}{cc} \cos x_3 & \sin x_3 \\ -\sin x_3 & \cos x_3 \end{array}\right)\right) \quad (32)$$

is universal covering epimorphism $A^+(\mathbb{R}) \times (\mathbb{R},+) \to Sol(2) \times SO(2)$.

The standard left-invariant sub-Riemannian structure on $A^+(\mathbb{R}) \times (\mathbb{R},+)$ is defined in [3] by the orthonormal frame $\Delta = span\{e_2, e_1 + e_3\}$. Then there is unique sub-Riemannian structure on $Sol(2) \times SO(2)$ such that F is a local isometry; it is defined by the orthonormal frame $\overline{\Delta} = span\{\overline{e_2}, \overline{e_1} + \overline{e_3}\}$, where

$$\overline{e_1} = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \quad \overline{e_2} = \left(\begin{array}{cc} -1/2 & 0 \\ 0 & 1/2 \end{array}\right), \quad \overline{e_3} = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right). \quad (33)$$

Now we follow [3]. Let $a = e^{-x_2}$ and $b = x_1$ in the second matrix of (31). The subgroup $A^+(\mathbb{R})$ is diffeomorphic to the half-plane $\{(a,b) \in \mathbb{R}^2, a > 0\}$, which is desrcibed in the standard polar coordinates as $\{(\rho,\theta) | \rho > 0, -\pi/2 < \theta < \pi/2\}$.

Theorem 4

[3]. The diffeomorphism $\Psi : A^+(\mathbb{R}) \times S^1 \to SL(2,\mathbb{R})$ defined by

$$\Psi(\rho,\theta,\varphi) = \frac{1}{\sqrt{\rho\cos\theta}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \rho\sin(\theta-\varphi) & \rho\cos(\theta-\varphi) \end{pmatrix}, \tag{34}$$

where $(\rho,\theta) \in A^+(\mathbb{R})$ and $\varphi \in S^1$, is a global sub-Riemannian isometry.

Remark 1

Using the above locally isometric covering F, we can and will understand Ψ as the global isometry between $Sol(2) \times SO(2)$ and $SL(2,\mathbb{R})$ supplied with sub-Riemannian metrics defined by the same frame $\overline{\Delta}$.

Corollary 4

$A^+(\mathbb{R}) \times (\mathbb{R}, +)$ with sub-Riemannian metric, defined by the frame Δ, is isometric to the universal covering $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ with sub-Riemannian metric such that the natural universal covering epimorphism of $\widetilde{SL(2,\mathbb{R})}$ onto $SL(2,\mathbb{R})$ with sub-Riemannian metric, defined by the frame $\overline{\Delta}$, is a local isometry.

Proposition 3

The global isometry Ψ in the sense of Remark 1 is the Iwasawa diffeomorphism of $Sol(2) \times SO(2)$ onto $SL(2,\mathbb{R})$ of the view $(n\overline{a}, k) \in NA \times SO(2) \to n\overline{a}k \in NAK = SL(2,\mathbb{R})$, where

$$n = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \overline{a} = \begin{pmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{pmatrix}, k = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

$a = \rho\cos\theta, b = \rho\sin\theta.$

Доказательство.

One needs simply to check that $n\overline{a}k$ is equal to the matrix in (34). □

Remark 2

Notice that $n = \exp(t\tilde{e}_1)$, $\overline{a} = \exp(s\overline{e_2})$, where $\tilde{e}_1 = (\overline{e}_1)^T$, ^{T} is the sign of transposition, $b = t$, and $a^{1/2} = e^{s/2}$. Also $[\tilde{e}_1, \overline{e_2}] = -\tilde{e}_1$.

THANK YOU VERY MUCH FOR ATTENTION!