# On the spetrum and complexity of Cayley graphs on a dihedral group

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Combinatorial-computational methods
of algebra and logics
devoted to the memory of Vitalij Anatolevich Romankov
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# Dedication

Devoted to the memory of our friend and colleague
**Vitalij Anatolevich Romankov.**
He was a **True man, Mathematician and Teacher.**

# Content of the lecture

1. Content
   - Introduction
   - Basic Definitions
   - Spectrum of a Cayley graph on the dihedral group
   - Arithmetical properties of complexity for the graph $\mathcal{D}_n$
   - Asymptotic formulas for the number of spanning trees
   - Generating function for the number of spanning trees
   - Examples

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# Introduction

Let $G$ be a finite connected graph. The notion of the complexity of a graph can be defined in several different ways. One can consider the number of edges or vertices, the number of spanning trees or rooted spanning forests. All the above-mentioned values can be expressed in terms of the Laplacian spectrum of a graph. In particular, by the famous Kirchhoff Matrix-Tree Theorem the number of spanning trees in a connected graph is equal to the product of all non-zero eigenvalues of its Laplacian matrix divided by the number of vertices.
The study of such invariants usually leads to the following question: **how to find the product of eigenvalues of the Laplacian matrix?** If the size (number of vertices) of a graph is small, it is an easy task. However, the most interesting cases involve the family of graphs with increasing number of vertices. The direct calculation of this product becomes tedious and unmanageable when the number of vertices $n$ of the graph tends to infinity. To solve this problem, we use the techniques developed in previous papers by the authors.

As a result, one can find a closed formula which is the product of a bounded number of factors, each given by the $n$-th Chebyshev polynomial of the first kind evaluated at the roots of some polynomial of prescribed degree. This paves the way to investigate arithmetical properties and asymptotics.
The complexity of a graph plays an important role in statistic physics, where the graphs with arbitrarily large number of vertices are considered. With increasing number of vertices the structure of the Laplacian characteristic polynomial becomes quite complicated. In this case, the most interesting invariants are given by their asymptotics.
The aim of the present report is to produce explicit analytic formulas for the number of spanning trees in a Cayley graph $\mathcal{D}_n$, see (1) for the definition, on a dihedral group. Such formulas help in the investigation of number–theoretical properties and asymptotics of several spectral invariants of the graph. This research expands series of publications by various authors on the complexity of circulant graphs. Note that the circulant graph is a Cayley graph on a cyclic group.
The methods laid out in the present report can be equally used to find explicit formulas for the number of rooted spanning forests and Kirchhoff index.

# Definition

Consider a finite connected graph $G$ with possibly multiple edges, but without loops. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$ respectively.

A *tree* is a connected undirected graph without cycles. A *spanning tree* in a graph $G$ is a subgraph that is a tree and contains all the vertices of $G$.

# Definition

Given $u, v \in V(G)$, we write $a_{uv}$ as the number of edges between vertices $u$ and $v$. The matrix $A = A(G) = \{a_{uv}\}_{u,v \in V(G)}$ is called *the adjacency matrix* of the graph $G$. The degree $d_v$ of a vertex $v \in V(G)$ is defined by $d_v = \sum_{u \in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d_v$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph $G$.
In what follows, we denote by $I_n$ the identity matrix of order $n$.
We say that an $n \times n$ matrix is *circulant,* denoted by $circ(a_0, a_1, \ldots, a_{n-1})$, if it is of the form

$$circ(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

# Definition

Let $D$ be a group, and let $S$ be a subset of $D$, which doesn't contain the identity element 1. The Cayley digraph associated with $(D, S)$ is then defined as the directed graph with the set of vertices $D$ and the set of edges

$$\{(g, h) : g, h \in D, gh^{-1} \in S\}.$$

The Cayley graph depends on the choice of a generating set $S$, and is connected if and only if $S$ generates $D$ (i.e., the set $S$ are group generators of $D$). We deal with undirected graphs, and always assume that $S = S^{-1}$, where $S^{-1} = \{s : s^{-1} \in S\}$.
Let $\mathbb{D}_n = \langle a, b | a^2 = 1, b^n = 1, (a\, b)^2 = 1 \rangle$ be dihedral group of order $2n$. We arrange the elements of the group $\mathbb{D}_n$ as
$V = \{1, b, \ldots, b^{n-1}, a, ba, \ldots, b^{n-1}a\}$ and consider the Cayley graph

$$\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t}) \qquad (1)$$

with the generating set $S = \{b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t}\}$ for some integers $\beta_1, \beta_2, \cdots$ , and $\gamma_1, \gamma_2, \cdots$ . We suppose that $\mathbb{D}_n$ acts on $V$ by the rule: $g \in \mathcal{D}_n$ sends a vertex $v \in V$ to the vertex $vg$.

# Definition

Then the set of oriented edges of $\mathcal{D}_n$ can be describe as follows. Given $j \in \{\pm\beta_1, \pm\beta_2, \ldots, \pm\beta_s\}$ there is an edge $b^k \xrightarrow{b^j} b^{k+j}$ and an edge $b^k a \xrightarrow{b^j} b^{k-j} a$ for any $k = 0, 1, \ldots, n-1$; given $j \in \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ there is an edge $b^k \xrightarrow{ab^j} b^{k-j} a$ and an edge $b^k a \xrightarrow{ab^j} b^{k+j}$ for any $k = 0, 1, \ldots, n-1$. Noting that $(ab^j)^{-1} = b^{-j} a = ab^j$, we have $S = S^{-1}$. Hence $\mathcal{D}_n$ is an undirected graph.

We always restrict to the case $0 < \beta_1 < \beta_2 < \cdots < \beta_s < \frac{n}{2}$ and $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_t \leq n-1$. We suppose that $s \geq 0$ and $t \geq 1$. Then the graph $\mathcal{D}_n$ has no loops and multiple edges.

# Auxiliary results

Now we introduce a necessary and sufficient condition for the connectedness of the graph $\mathcal{D}_n$.

**Lemma 1**

The graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t})$ is connected if and only if $\gcd(n, \beta_j, 1 \le j \le s, \gamma_j - \gamma_k, 1 \le j < k \le t)$ is equal to 1.

# Auxiliary results

The adjacency matrix of the graph $\mathcal{D}_n$ is given by the $2n \times 2n$ block matrix

$$A = \begin{pmatrix} \sum_{j=1}^{s}(T_n^{\beta_j} + T_n^{-\beta_j}) & \sum_{j=1}^{t} T_n^{-\gamma_j} \\ \sum_{j=1}^{t} T_n^{\gamma_j} & \sum_{j=1}^{s}(T_n^{\beta_j} + T_n^{-\beta_j}) \end{pmatrix},$$

where $T_n = circ(0, 1, 0, \ldots, 0)$. The corresponding degree matrix is $D = \begin{pmatrix} (2s+t)I_n & 0 \\ 0 & (2s+t)I_n \end{pmatrix}$, where $I_n$ is the $n \times n$ identity matrix. Since the Laplacian of $\mathcal{D}_n$ is $L = D - A$, we have

$$L = \begin{pmatrix} (2s+t)I_n - \sum_{j=1}^{s}(T_n^{\beta_j} + T_n^{-\beta_j}) & -\sum_{j=1}^{t} T_n^{-\gamma_j} \\ -\sum_{j=1}^{t} T_n^{\gamma_j} & (2s+t)I_n - \sum_{j=1}^{s}(T_n^{\beta_j} + T_n^{-\beta_j}) \end{pmatrix}.$$

# Spectrum of a Cayley graph on the dihedral group

Let $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t})$ be the Cayley graph on a dihedral group $\mathbb{D}_n$. We introduce the following Laurent polynomials

$$\mathcal{A}(z) = 2s + t - \sum_{i=1}^{s}(z^{\beta_i} + z^{-\beta_i}),\ \mathcal{B}(z) = -\sum_{i=1}^{t} z^{\gamma_i}$$

and

$$P(z) = \mathcal{A}(z)\mathcal{A}(z^{-1}) - \mathcal{B}(z)\mathcal{B}(z^{-1}).$$

We will refer to $P(z)$ as *the Laurent polynomial associated with the graph* $\mathcal{D}_n$. Then the Laplacian of the graph $\mathcal{D}_n$ is given by the following $2n \times 2n$ block matrix $L = \begin{pmatrix} \mathcal{A}(T) & \mathcal{B}(T^{-1}) \\ \mathcal{B}(T) & \mathcal{A}(T^{-1}) \end{pmatrix}$, where $T = circ(0, 1, 0, \ldots, 0)$.

# Spectrum of a Cayley graph on the dihedral group

Recall that $T$ is conjugated to $\mathbb{T} = diag(1, \varepsilon, \ldots, \varepsilon^{n-1})$, where $\varepsilon = \varepsilon_n = \exp(2\pi \mathrm{i}/n)$. So that $L$ is conjugated to the matrix $\mathbb{L} = \begin{pmatrix} \mathcal{A}(\mathbb{T}) & \mathcal{B}(\mathbb{T}^{-1}) \\ \mathcal{B}(\mathbb{T}) & \mathcal{A}(\mathbb{T}^{-1}) \end{pmatrix}$. The spectra of $L$ and $\mathbb{L}$ are the same. It can be found by solving the system of linear equations

$$\begin{cases} \mathcal{A}(\mathbb{T})\mathbf{x} + \mathcal{B}(\mathbb{T}^{-1})\mathbf{y} = \lambda \mathbf{x} \\ \mathcal{B}(\mathbb{T})\mathbf{x} + \mathcal{A}(\mathbb{T}^{-1})\mathbf{y} = \lambda \mathbf{y}, \end{cases}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0})$. Since the matrix $\mathbb{T}$ is diagonal, the system of equations splits into $n$ scalar linear systems

$$\begin{cases} \mathcal{A}(\varepsilon^j)x + \mathcal{B}(\varepsilon^{-j})y = \lambda x \\ \mathcal{B}(\varepsilon^j)x + \mathcal{A}(\varepsilon^{-j})y = \lambda y, \end{cases}$$

where $j = 0, 1 \ldots, n-1$. Hence, $\lambda$ is a root of the quadratic equation

$$\lambda^2 - (\mathcal{A}(\varepsilon^j) + \mathcal{A}(\varepsilon^{-j}))\lambda + \mathcal{A}(\varepsilon^j)\mathcal{A}(\varepsilon^{-j}) - \mathcal{B}(\varepsilon^j)\mathcal{B}(\varepsilon^{-j}) = 0.$$

# Spectrum of a Cayley graph on the dihedral group

The solutions of this equation are
$\lambda_{j,1} = \Re(\mathcal{A}(\varepsilon^j)) + \sqrt{-\Im(\mathcal{A}(\varepsilon^j))^2 + |\mathcal{B}(\varepsilon^j)|^2}$ and
$\lambda_{j,2} = \Re(\mathcal{A}(\varepsilon^j)) - \sqrt{-\Im(\mathcal{A}(\varepsilon^j))^2 + |\mathcal{B}(\varepsilon^j)|^2}$. We note that L. Babai (1979) found similar formulas for eigenvalues for the adjacency matrices of $\mathcal{D}_n$ by making use of the representation theory for finite groups. The corresponding eigenvectors of the operator $\mathbb{L}$ are $u_{j,1} = \mathcal{B}(\varepsilon^j)\mathbf{e}_{j+1} + (\lambda_{j,1} - \mathcal{A}(\varepsilon^j))\mathbf{e}_{j+1}$ and $u_{j,2} = \mathcal{B}(\varepsilon^j)\mathbf{e}_{j+1} + (\lambda_{j,2} - \mathcal{A}(\varepsilon^j))\mathbf{e}_{j+1}$, where $\mathbf{e}_j$ is the $j$-th basic vector in the $\mathbb{C}^{2n}$.

For the special case $j = 0$, we have $\lambda^2 - 2\mathcal{A}(1)\lambda + \mathcal{A}(1)^2 - \mathcal{B}(1)^2 = 0$. Hence

$$\lambda_{0,1} = \mathcal{A}(1) + \mathcal{B}(1) = 0,\ \lambda_{0,2} = \mathcal{A}(1) - \mathcal{B}(1) = 2\mathcal{A}(1).$$

We note also that $\lambda_{j,1}\lambda_{j,2} = \mathcal{A}(\varepsilon^j)\mathcal{A}(\varepsilon^{-j}) - \mathcal{B}(\varepsilon^j)\mathcal{B}(\varepsilon^{-j}) = P(\varepsilon^j)$. We suppose that the graph $\mathcal{D}_n$ is connected. Then all the Laplacian eigenvalues, except of $\lambda_{0,1} = 0$, are non-zero. They are

$$\lambda_{0,2},\ \lambda_{j,1},\ \lambda_{j,2},\ j = 1, 2, \ldots, n-1.$$

# Spectrum of a Cayley graph on the dihedral group

By the Kirchhoff's theorem we have

$$\tau(n) = \frac{\lambda_{0,2}}{2n} \prod_{j=1}^{n-1} \lambda_{j,1}\lambda_{j,2} = \frac{\mathcal{A}(1)}{n} \prod_{j=1}^{n-1} P(\varepsilon^j).$$

Since $\mathcal{A}(1) = t$, we get the following result.

## Theroem 1

Let $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t})$ be a Cayley graph on the dihedral group $\mathbb{D}_n$. Then the number of spanning trees of $\mathcal{D}_n$ is given by the formula

$$\tau(n) = \frac{t}{n} \prod_{j=1}^{n-1} P(\varepsilon_n^j),$$

where $\varepsilon_n = \exp(2\pi \mathrm{i}/n)$, $P(z) = \mathcal{A}(z)\mathcal{A}(z^{-1}) - \mathcal{B}(z)\mathcal{B}(z^{-1})$, $\mathcal{A}(z) = 2s + t - \sum_{i=1}^{s}(z^{\beta_i} + z^{-\beta_i})$ and $\mathcal{B}(z) = -\sum_{i=1}^{t} z^{\gamma_i}$.

# Properties of the associated polynomial $P(z)$

To investigate more deep properties of the associated polynomial $P(z)$ we have the following two lemmas.

**Lemma 2**

Let graph $\mathcal{D}_n$ be connected. Then $P(1) = 0, P'(1) = 0$ and $P''(1) < 0$.

**Lemma 3**

Suppose that the numbers $\{\beta_j,\ 1 \le j \le s,\ \gamma_j - \gamma_k,\ 1 \le j < k \le t\}$ are relatively prime. Then for any $\varphi \in \mathbb{R}$, we have $P(e^{\mathrm{i}\varphi}) \ge 0$. Furthermore, $P(e^{\mathrm{i}\varphi}) = 0$ if and only if $e^{\mathrm{i}\varphi} = 1$.

# Counting spanning trees

One of the main results of this report are the two following theorems.

## Theorem 2

Let $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\beta_1}, \ldots, b^{\beta_s}, ab^{\gamma_1}, \ldots, ab^{\gamma_t})$ be a Cayley graph on the group $\mathbb{D}_n$ and $P(z)$ is the associated Laurent polynomial for $\mathbb{D}_n$. Then the number of spanning trees $\tau(n)$ in the graph $\mathcal{D}_n$ is given by the formula

$$\tau(n) = \frac{n\,t\,|\eta|^n}{q} \prod_{\substack{P(z)=0 \\ z \neq 1}} |z^n - 1|,$$

where the product is taken over all the roots different from 1 of be the associated Laurent polynomial $P(z)$, $\eta$ is the leading coefficient of $P(z)$, and $q = 2t \sum_{j=1}^{s} \beta_j^2 + \sum_{1 \le j < k \le t} (\gamma_j - \gamma_k)^2$.

# Counting spanning trees

Let us outline the proof of Theorem 2. By Theorem 1 we already have

$$\tau(n) = \frac{t}{n}\prod_{j=1}^{n-1} P(\varepsilon_n^j). \qquad (2)$$

Denote by $\eta$ the leading coefficient by $P(z)$. Recall that $P(z) = A(z)A(z^{-1}) - B(z)B(z^{-1})$, where

$$A(z) = 2s + t - \sum_{i=1}^{s}(z^{\beta_i} + z^{-\beta_i}),\ B(z) = -\sum_{i=1}^{t} z^{\gamma_i}$$

Since $P(z) = P(1/z)$ we can present polynomial in the form

$$P(z) = \eta\, z^{-r} + a_1 z^{-r+1} + \ldots + a_r + \ldots + a_1 z^{r-1} + \eta\, z^r,$$

for some $r \geq 0$. We still don't know geometrical meaning of the leading term $\eta\, z^r$ of $P(z)$.

# Counting spanning trees

To continue the proof we replace the Laurent polynomial $P(z)$ by $\widetilde{P}(z) = \frac{z^r}{\eta}P(z)$. Then $\widetilde{P}(z)$ is a monic polynomial of the degree $2r$ with the same roots as $P(z)$. We note that

$$\prod_{j=1}^{n-1}\widetilde{P}(\varepsilon_n^j) = \frac{\varepsilon_n^{\frac{(n-1)n}{2}r}}{\eta^{n-1}}\prod_{j=1}^{n-1}P(\varepsilon_n^j) = \frac{(-1)^{r(n-1)}}{\eta^{n-1}}\prod_{j=1}^{n-1}P(\varepsilon_n^j). \quad (3)$$

By Lemma 2 the polynomial $\widetilde{P}(z)$ has two roots equal to 1 and all the other roots different from 1. Also, we recognize the complex numbers $\varepsilon_n^j$, $j = 1, \ldots, n-1$ as the roots of polynomial $\frac{z^n-1}{z-1}$. By the basic properties of resultant we have

$$\prod_{j=1}^{n-1}\widetilde{P}(\varepsilon_n^j) = \mathrm{Res}(\widetilde{P}(z), \frac{z^n-1}{z-1}) = \mathrm{Res}(\frac{z^n-1}{z-1}, \widetilde{P}(z)) = \prod_{z:\,\widetilde{P}(z)=0}\frac{z^n-1}{z-1}$$

$$= \prod_{z:\,P(z)=0}\frac{z^n-1}{z-1} = (\lim_{z\to1}\frac{z^n-1}{z-1})^2\prod_{\substack{P(z)=0\\ z\neq1}}\frac{z^n-1}{z-1} = n^2\prod_{\substack{P(z)=0\\ z\neq1}}\frac{z^n-1}{z-1}. \quad (4)$$

# Counting spanning trees

Combine (2), (3), and (4) we have the following formula for the number of spanning trees

$$\tau(n) = (-1)^{r(n-1)}\eta^{n-1} n\, t \prod_{\substack{P(z)=0 \\ z\neq 1}} \frac{z^n - 1}{z - 1}. \tag{5}$$

To finish the proof we need to evaluate the product $\prod\limits_{\substack{P(z)=0 \\ z\neq 1}} (z-1)$. The last term is equal to $\frac{\widetilde{P}''(1)}{2} = \frac{1}{2}(\frac{z^r}{\eta}P(z))''_{z=1} = \frac{P''(1)}{2\eta} = -\frac{q}{\eta}$. As a result, we get

$$\tau(n) = \frac{(-1)^{r(n-1)+1} n\, t\, \eta^n}{q} \prod_{\substack{P(z)=0 \\ z\neq 1}} (z^n - 1). \tag{6}$$

Since $\tau(n)$ is a positive integer, the statement of theorem follows.

# Counting spanning trees

For each Laurent polynomial $P(z)$, satisfying the property $P(z) = P(1/z)$ we can introduce ordinary polynomial $Q(z)$ which is related to $P(z)$ in tyhe following way $P(z) = Q(\frac{z+z^{-1}}{2})$. Also, $Q(w) = P(w + \sqrt{w^2 - 1})$.
Denote by $T(n, x) = \cos(n \arccos(x))$ the $n$-th Chebyshev polynomial of the first kind. The following equality is known $T(n, \frac{z+z^{-1}}{2}) = \frac{z^n+z^{-n}}{2}$. Because of this property we refer to polynomial $Q(w)$ as *the Chebyshev transform* of $P(z)$. It is easier to deal with $Q(w)$ since it is an ordinary polynomial with degree twice less than $P(z)$.
By Lemmas 2 and 3, the roots of polynomials $P(z)$ and $Q(w)$ are $1, 1, z_1, 1/z_1, \ldots, z_{r-1}, 1/z_{r-1}$, $z_j \neq 1$ and $1 \neq w_j = \frac{1}{2}(z_j + z_j^{-1})$, $j = 1, \ldots, r-1$, respectively. Also $T(n, w_j) = \frac{z_j^n+z_j^{-n}}{2}$. Hence,

$$\prod_{\substack{P(z)=0 \\ z\neq 1}} (z^n - 1) = \prod_{j=1}^{r-1} (z_j^n - 1)(z_j^{-n} - 1) = (-1)^{r-1} \prod_{j=1}^{r-1} (2T(n, w_j) - 2).$$

Substituting the latter in Theorem 2 we get the following theorem.

# Counting spanning trees

**Theorem 3**

Let $P(z)$ be the associated Laurent polynomial of the Cayley graph $\mathcal{D}_n$. Denote by $Q(w)$ the Chebyshev transform of $P(z)$ and let $r$ be the degree of polynomial $Q(w)$. The number of spanning trees $\tau(n)$ in the graph $\mathcal{D}_n$ is given by the formula

$$\tau(n) = \frac{n\, t|\eta|^n}{q} \prod_{p=1}^{r-1} |2T(n, w_p) - 2|,$$

where $w_p$, $p = 1, 2, \ldots, r-1$ are different from 1 roots of the algebraic equation $Q(w) = 0$, and $T(n, w)$ is the Chebyshev polynomial of the first kind and $\eta$ and $q$ are the same as above.

# Arithmetical properties of complexity for the graph $\mathcal{D}_n$

The main result of this section is the following theorem.

**Theorem 4**

Let $\tau(n)$ be the number of spanning trees for the graph

$$\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\beta_1}, b^{\beta_2}, \ldots, b^{\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t}).$$

Denote by $\beta_{odd}$ and $\gamma_{odd}$ the number of odd numbers in the sequences $\beta_1, \beta_2, \ldots, \beta_s$ and $\gamma_1, \gamma_2, \ldots, \gamma_t$ respectively. Also, denote by $\gamma_{even}$ the number of even numbers in the sequence $\gamma_1, \gamma_2, \ldots, \gamma_t$. Let $\delta$ be the square free part of the integer $\xi = (2\beta_{odd} + \gamma_{odd})(2\beta_{odd} + \gamma_{even})$. Then there exists an integer sequence $a(n)$ such that

$1^0$ $\tau(n) = n\, t\, a(n)^2$, if $n$ is odd;

$2^0$ $\tau(n) = n\, t\, \delta\, a(n)^2$, if $n$ is even.

Let show the sketch of the proof.
We consider associated polynomial $P(z)$ for the respective graph $\mathcal{D}_n$. Now we will express the value $P(-1)$ through basic parameters of $\mathcal{D}_n$. Denote by $\beta_{even}$ and $\beta_{odd}$ the number of even numbers and odd numbers in the sequences $\beta_1, \beta_2, \ldots, \beta_s$ respectively. Also, denote by $\gamma_{even}$ and $\gamma_{odd}$ the number of even numbers and odd numbers in the sequence $\gamma_1, \gamma_2, \ldots, \gamma_t$. It easy to see that $s = \beta_{even} + \beta_{odd}$ and $t = \gamma_{even} + \gamma_{odd}$. Since $P(z) = A(z)A(z^{-1}) - B(z)B(z^{-1})$ by direct calculations we get

$$P(-1) = A(-1)^2 - B(-1)^2 = 4(2\beta_{odd} + \gamma_{even})(2\beta_{odd} + \gamma_{odd}).$$

As consequence, we have $P(-1) = 4\xi = \delta(2\omega)^2$, where $\delta$ is square free part of $\xi$ and $\omega$ is some integer.

By formula (2) we have $n\,\tau(n) = t\prod_{j=1}^{n-1}\lambda_{j,1}\lambda_{j,2}$. Note that $\lambda_{j,1}\lambda_{j,2} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{n-j,1}\lambda_{n-j,2}$. Define $c(n) = \prod\limits_{j=1}^{\frac{n-1}{2}}\lambda_{j,1}\lambda_{j,2}$, if $n$ is odd and $d(n) = \prod\limits_{j=1}^{\frac{n}{2}-1}\lambda_{j,1}\lambda_{j,2}$, if $n$ is even. Following [?] we note that each algebraic number $\lambda_{i,j}$ comes into both products $\prod_{j=1}^{(n-1)/2}\lambda_{j,1}\lambda_{j,2}$ and $\prod_{j=1}^{n/2-1}\lambda_{j,1}\lambda_{j,2}$ with all of its Galois conjugate elements. Therefore, both products $c(n)$ and $d(n)$ are integer numbers. Moreover, if $n$ is even we get $\lambda_{\frac{n}{2},1}\lambda_{\frac{n}{2},2} = P(-1)$. Now, we have

1° $n\,\tau(n) = t\,c(n)^2$ if $n$ is odd,

2° $n\,\tau(n) = t\,P(-1)\,d(n)^2 = 4t\,\delta\,\omega^2\,d(n)^2$ if $n$ is even.

# Arithmetical properties of complexity for the graph $\mathcal{D}_n$

Note that using formula (5) from the proof of Theorem 2 we conclude that $\frac{\tau(n)}{n\,t}$ is an integer. Indeed, since $\widetilde{P}(z)$ and $P(z)$ share the roots we have

$$\tau(n) = (-1)^{r(n-1)}\eta^{n-1} n\,t \prod_{\substack{\widetilde{P}(z)=0 \\ z\neq 1}} \frac{z^n - 1}{z - 1}.$$

The last product is equal to resultant of two integer polynomials $\frac{\widetilde{P}(z)}{(z-1)^2}$ and $\frac{z^n-1}{z-1}$ and, hence, it is an integer number. So that $\frac{\tau(n)}{n\,t}$ is also an integer. We get

1° $\frac{\tau(n)}{n\,t} = (\frac{c(n)}{n})^2$ if $n$ is odd,

2° $\frac{\tau(n)}{n\,t} = \delta\,(\frac{2\omega\,d(n)}{n})^2$ if $n$ is even.

As $\frac{\tau(n)}{n\,t}$ is an integer and $\delta$ is square free, all squared rational numbers in 1° and 2° are integers. We set $a(n) = \frac{c(n)}{n}$ if $n$ is odd and $a(n) = \frac{2\omega\,d(n)}{n}$ if $n$ is even. This proves the theorem.

# Asymptotic formulas for the number of spanning trees

In this section devoted to the asymptotics for the number of spanning trees in the graph

$$\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t}).$$

To do this we suppose that parameters $\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t$ are fixed, and the inequalities $0 < \beta_1 < \beta_2 < \cdots < \beta_s < \frac{n}{2}$ and $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_t \le n-1$ hold for all sufficiently large values $n$. We suppose also that graphs $\mathcal{D}_n$ are connected.

# Asymptotic formulas for the number of spanning trees

## Theorem 5

Let $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm\beta_1}, b^{\pm\beta_2}, \ldots, b^{\pm\beta_s}, ab^{\gamma_1}, ab^{\gamma_2}, \ldots, ab^{\gamma_t})$ be an infinite family of connected graphs. Then the asymptotic behaviour for the number of spanning trees $\tau(n)$ for the graph $\mathcal{D}_n$ is given by the formula

$$\tau(n) \sim \frac{n\,t}{q} A^n,\ n \to \infty,$$

where $A = \exp\left(\int\limits_0^1 \log P(e^{2\pi \mathrm{i} t})\mathrm{d}t\right)$ and $q = 2t\sum_{j=1}^{s}\beta_j^2 + \sum_{1\le j<k\le t}(\gamma_j - \gamma_k)^2.$

# Asymptotic formulas for the number of spanning trees

By Theorem 3 we have $\tau(n) = \frac{n\,t\,|\eta|^n}{q}\prod_{j=1}^{r-1}|2T(n,w_j)-2|$, where $w_j,\ j=1,2,\ldots,r-1$ are roots, different from 1, of the Chebyshev transform of $P(z)$.

By Lemma 3, $T(n,w_j)=\frac{z_j^n+z_j^{-n}}{2}$, where the $z_j$ and $1/z_j$ are roots of the polynomial $P(z)$ with the property $|z_j|\neq 1,\ j=1,2,\ldots,r-1$. Replacing $z_j$ by $1/z_j$, if it is necessary, we can assume that $|z_j|>1$ for all $j=1,2,\ldots,r-1$. Then $T(n,w_j)\sim\frac{1}{2}z_j^n$ and $|2T(n,w_j)-2|\sim|z_j|^n$ as $n\to\infty$. Hence

$$\frac{n\,t|\eta|^n}{q}\prod_{j=1}^{r-1}|2T_n(w_j)-2|\sim\frac{n\,t|\eta|^n}{q}\prod_{j=1}^{r-1}|z_j|^n=\frac{n\,t}{q}|\eta|^n\prod_{\substack{P(z)=0,\\|z|>1}}|z|^n=\frac{n\,t\,A^n}{q},$$

where $A=|\eta|\prod_{P(z)=0,\ |z|>1}|z|$ is the Mahler measure of the polynomial $P(z)$.

By definition of Mahler measure, we have $A=\exp\left(\int_0^1\log|P(e^{2\pi it})|\mathrm{d}t\right)$.

The theorem is proved.

# Generating function for the number of spanning trees

In this section, our aim is to prove the following result.

**Theorem 6**

Let $\tau(n)$ be the number of spanning trees in the graph $\mathcal{D}_n$. Then $F(x) = \sum_{n=1}^{\infty} \tau(n)x^n$ is a rational function with integer coefficients. Moreover, $F(\eta x) = F(\frac{1}{\eta x})$, where $\eta$ is the leading coefficient of the associated polynomial $P(z)$. The latter allows to represent $F(x)$ as a rational function of $u = \frac{1}{2}(\eta x + \frac{1}{\eta x})$.

# Generating function for the number of spanning trees

The proof of Theorem 6 is based on the following proposition earlier proved by authors.

## Proposition 1

Let $R(z)$ be a degree $2s$ polynomial with integer coefficients. Suppose that all the roots of the polynomial $R(z)$ are $\xi_1, \xi_2, \ldots, \xi_{2s-1}, \xi_{2s}$. Then

$$F(x) = \sum_{n=1}^{\infty} \left( n \prod_{j=1}^{2s} (\xi_j^n - 1) \right) x^n$$

is a rational function with integer coefficients.
Moreover, if $\xi_{j+s} = \xi_j^{-1}$, $j = 1, 2, \ldots, s$, then $F(x) = F(1/x)$.

Proof of Theorem 6. By formula (6) we have

$$F(x) = \sum_{n=1}^{\infty} \tau(n)x^n = \sum_{n=1}^{\infty} \Big( \frac{(-1)^{r(n-1)+1} n\, t\, \eta^n}{q} \prod_{\substack{P(z)=0 \\ z \neq 1}} (z^n - 1) \Big) x^n.$$

Since all the roots of $P(z)$, different from 1, are $z_1, 1/z_1, \ldots, z_{r-1}, 1/z_{r-1}$, we can rewrite the latter as

$$F(x) = \frac{(-1)^{-r+1} t}{q} \sum_{n=1}^{\infty} \Big( n \prod_{j=1}^{r-1} (z_j^n - 1)(z_j^{-n} - 1) \Big) ((-1)^r \eta\, x)^n.$$

Since $t$ and $q$ are rational numbers, by Proposition 1, $F(x)$ is a rational function with integer coefficients satisfying $F((-1)^r \eta\, x) = F(\frac{1}{(-1)^r \eta\, x})$. Hence, $F(\eta\, x) = F(\frac{1}{\eta\, x})$.

# Example 1. Prism graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm 1}, a)$.

**1°. The number of spanning trees.** The associated Laurent polynomial and its Chebyshev transform are

$$P(z) = z^{-2} - 6z^{-1} + 10 - 6z + z^2 \text{ and } Q(w) = 4(w-2)(w-1).$$

Here $t = 1$, $\eta = 1$, $q = 2$. Hence, by Theorems 2 and 3 we have

$$\tau(n) = \frac{n\,t\,\eta^n}{q}(2T(n,2) - 2) = n(T(n,2) - 1).$$

This coincides with the well-known result.

**2°. The asymptotics of $\tau(n)$.** By Theorem 5, $\tau(n) \cong \frac{n}{2}A^n$, where $A = 2 + \sqrt{3}$.

# Example 1. Prism graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm 1}, a)$.

**3°. The generating function of $\tau(n)$.** Theorem 6 gives

$$F(x) = \sum_{n=1}^{\infty} \tau(n) x^n = \frac{-3 + u + u^2}{2(-2+u)^2(-1+u)},$$

where $u = \frac{1}{2}(x + \frac{1}{x})$.

**4°. Divisibility by squares.** To see the divisibility by squares, consider a few terms of generating function

$$F(x) = x + 12x^2 + 75x^3 + 384x^4 + 1805x^5 + 8100x^6 + 35287x^7 + 150528x^8 + \ldots.$$

By Theorem 4, we have
$\xi = (2\beta_{odd} + \gamma_{odd})(2\beta_{odd} + \gamma_{even}) = (2 \cdot 1)(2 \cdot 1 + 1) = 6$. Hence, $\delta = 6$. So that there exists an integer sequence $a(n)$ such that $\tau(u) = n\, a(n)^2$ if $n$ is odd and $\tau(u) = 6n\, a(n)^2$ if $n$ is even.

# Example 2. Dihedral graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm 1}, b^{\pm 2}, ab, ab^3, ab^5)$.

$1^\circ$. **The number of spanning trees.** By Theorem 2

$$\tau(n) = \frac{n\,t\,\eta^n}{q}|(-2)^n - 1| \cdot |(-1/2)^n - 1| \cdot |(4-\sqrt{15})^n - 1| \cdot |(4+\sqrt{15})^n - 1|,$$

where $t = 3$, $\eta = 2$, $q = 54$. Equivalently, by Theorem 3 we get

$$\tau(n) = \frac{n\,2^n}{18}|2T(n, -\frac{5}{4}) - 2| \cdot |2T(n, 4) - 2|.$$

$2^\circ$. **The asymptotics of $\tau(n)$.** By Theorem 5, we have $\tau(n) \cong \frac{n}{18}A^n$, where $A = 4(4+\sqrt{15})$.

# Example 2. Dihedral graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm 1}, b^{\pm 2}, ab, ab^3, ab^5)$.

**3°. The generating function of $\tau(n)$.** From Theorem 6, we get

$$F(x) = \sum_{n=1}^{\infty} \tau(n) x^n = \frac{R(x)}{S(x)}$$

where

$$R(x) = 6(-1745300 + 4540750u - 3003815u^2 + 346990u^3 + 171265u^4 -$$

$$-47660u^5 + 4840u^6 - 272u^7 + 16u^8)$$

and

$$S(x) = (2 + u)(8 + u)^2(-5 + 2u)^2(265 - 80u + 4u^2)^2.$$

Everywhere $u = \frac{1}{2}(2x + \frac{1}{2x})$.

# Example 2. Dihedral graph $\mathcal{D}_n = Cay(\mathbb{D}_n, b^{\pm 1}, b^{\pm 2}, ab, ab^3, ab^5)$.

4°. **Divisibility by squares.** To see the divisibility by squares we consider a few terms of generating function

$$F(x) = 3x + 60x^2 + 6561x^3 + 192000x^4 + 9149415x^5 + 315059220x^6 + \cdots .$$

By Theorem 4, we have
$\xi = (2\beta_{odd} + \gamma_{odd})(2\beta_{odd} + \gamma_{even}) = (2 \cdot 1 + 3)(2 \cdot 1 + 0) = 10$. Hence,
$\delta = 10$. So that there exists an integer sequence $a(n)$ such that
$\tau(u) = 3n\, a(n)^2$ if $n$ is odd and $\tau(u) = 30n\, a(n)^2$ if $n$ is even.