Commutative Poisson algebras from deformations of noncommutative algebras and non-Abelian Hamiltonian systems.

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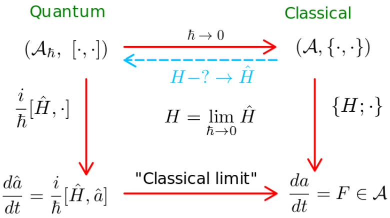
Outline

Classical limit and Deformation Quantisation



Deformation quantisation (deformation of the multiplication):

$$f \cdot g \longrightarrow f \star g = f \cdot g + \hbar (f,g)_1 + \hbar^2 (f,g)_2 + \cdots$$

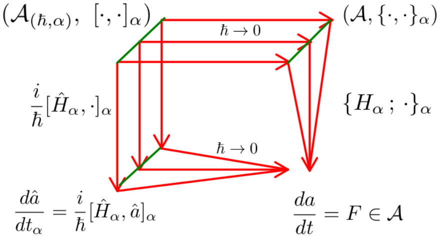
$$f \star g - g \star f = i\hbar \{f, g\} + \mathcal{O}(\hbar^2).$$

Issues:

- Canonical transformations – a choice of canonical variables.
- Associativity of the deformed non-commutative multiplication.
- Consistency of the algebra with the equations of motion for finite $\hbar$.
- Ordering of operators in the Hamiltonian and other observables.

Quantum and Poisson pencil

Quantum and Poisson pencils



Uncharted territories:

- Quantisation of systems admitting a multi-Hamiltonian structure.
- Can we define a non-deformation quantisation?

Problem of quantisation, quantisation ideals

Fact: Any finitely generated associative algebra can be realised as a quotient of a free algebra $\mathfrak{A}$ over an appropriate two sided ideal $\mathfrak{J}$.

In Algebraic Quantisation, the problem of quantisation of a free associative dynamical system (i.e. a derivation $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$) can be formulated as:

To find a two sided ideal $\mathfrak{J} \subset \mathfrak{A}$ such that

A. $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \quad \Leftrightarrow$ the derivation ∂_t induces a derivation of the quotient algebra $\mathfrak{A}/\mathfrak{J}$.

B. The quotient algebra $\mathfrak{A}/\mathfrak{J}$ has an additive basis of normally ordered monomials. In other words, we know how to change the order of any two variables.

An ideal $\mathfrak{J}$ satisfying the conditions A, B is called a quantisation ideal and the corresponding quotient algebra $\mathfrak{A}/\mathfrak{J}$ a quantum algebra.

Application to a classical dynamical system with commutative variables:

Step #0: To lift the dynamical system to a free algebra.

Problem of quantisation, quantisation ideals

The method was successfully applied to:
Volterra, Bogoyavlensky, Toda, Ablowitz-Ladik hierarchies (S.Carpentier, J.P.Wang, AVM),
Euler top and Zhukovsky-Volterra top (AVM, T.Skrypnyk)
Stationary KDV and Novikov's hierarchies (V.M.Buchstaber, AVM)

Example: Volterra type equation:

$$\frac{dx_\ell}{dt_2} = x_\ell x_{\ell+1}^2 - x_{\ell-1}^2 x_\ell + x_\ell^2 x_{\ell+1} - x_{\ell-2}x_{\ell-1}x_\ell + x_\ell x_{\ell+1}x_{\ell+2} - x_{\ell-1}x_\ell^2$$

on a free algebra $\mathfrak{A} = \langle \ldots, x_{-1}, x_0, x_1, \ldots \rangle$ ideal $\mathcal{I} \subset \mathfrak{A}$

$$\mathcal{I} = \langle x_n x_m - \omega_{n,m} x_m x_n\,;\ n > m \rangle.$$

The ∂_t–stability condition $\partial_t \mathcal{J} \subset \mathfrak{A}$ give rise a system of equations on the parameters $\omega_{n,m}$ which has two solutions

$$\mathcal{J}^{(1)}(\omega) = \langle x_{n+1}x_n - \omega x_n x_{n+1},\ x_n x_m - x_m x_n\,;\ |n-m| > 1 \rangle;$$

$$\mathcal{J}^{(2)}(\omega) = \langle x_{n+1}x_n - (-1)^n \omega x_n x_{n+1},\ x_n x_m + x_m x_n\,;\ |n-m| > 1 \rangle$$

On the quantum algebras $\mathcal{A}_\omega^{(1)} = \mathfrak{A}/\mathcal{J}^{(1)}(\omega)$ and $\mathcal{A}_\omega^{(2)} = \mathfrak{A}/\mathcal{J}^{(2)}(\omega)$ we have

$$\frac{dx_\ell}{dt_2} = \frac{1}{\omega^2 - 1}[\mathcal{H}, x_\ell], \quad \mathcal{H} = \sum_{k \in \mathbb{Z}} x_k^2 + x_{k+1}x_k + x_k x_{k+1}.$$

Motivating example: A simple system on the quantum plane

The quantum plane: $\mathcal{A}_q = \mathbb{C}(q)\langle x, y\rangle / \langle yx - qxy\rangle$. Hamiltonian:

$$\hat{H} = (y - qx)^2 = y^2 - q(q+1)xy + q^2x^2 \in \mathcal{A}_q$$

the Heisenberg equation

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] = xy^2 - qx^2y, \quad \frac{dy}{dt} = \frac{1}{q^2-1}[\hat{H}, y] = \cdots$$

In the classical limit $q = 1 + \nu \to 1$, $\mathcal{A}_q \to \mathcal{A} = \mathbb{C}[x, y]$, $(\nu = i\hbar)$

$$\{a, b\} = \lim_{\nu \to 0} \frac{1}{\nu}[a, b] = \left(\frac{\partial a}{\partial y}\frac{\partial b}{\partial x} - \frac{\partial b}{\partial y}\frac{\partial a}{\partial x}\right)xy, \quad a, b \in \mathbb{C}[x, y].$$

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] \to xy^2 - x^2y = \{H, x\}, \quad H = \frac{1}{2}(y-x)^2$$

In the limit: $q \to -1$, $\mathcal{A}_q \to \mathcal{A} = \mathbb{C}\langle x, y\rangle / \langle yx + xy\rangle$, and $\hat{H} \to (x^2 + y^2)$

$$\frac{dx}{dt} = \frac{1}{q^2-1}[\hat{H}, x] \to xy^2 + x^2y \overset{?!}{=} \{\mathbf{H}; x\}, \quad \mathbf{H} = ?!$$

Volterra type hierarchy

Volterra type system on the algebra $\mathcal{A}_q := \mathbb{C}(q)\langle x_i\,;\, i \in \mathbb{Z}\rangle / J_q$

$$J_q = \langle x_{i+1}x_i - (-1)^i q\, x_i x_{i+1},\; x_i x_j + x_j x_i\,;\; i, j \in \mathbb{Z},\; |i-j| > 1\rangle.$$

$$\frac{dx_\ell}{dt_2} = \frac{1}{q^2-1}[\hat{H}_2, x_\ell] = x_\ell x_{\ell+1}^2 - x_{\ell-1}^2 x_\ell + x_\ell^2 x_{\ell+1} - x_{\ell-2}x_{\ell-1}x_\ell + x_\ell x_{\ell+1}x_{\ell+2} - x_{\ell-1}x_\ell^2$$

$$\hat{H}_2 = \sum_{k\in\mathbb{Z}} \left(x_k^2 + (1 + (-1)^k q)x_k x_{k+1}\right)$$

There is a quantum hierarchy of symmetries (SC,AVM,JPW):

$$\frac{dx_\ell}{dt_{2m}} = \frac{1}{q^{2m}-1}[\hat{H}_{2m}, x_\ell], \quad [\hat{H}_{2m}, \hat{H}_{2n}] = 0.$$

There is a well defined limit $q \to 1$:

$$\mathcal{A}_q \to \mathcal{A} = \mathbb{C}\langle \ldots, x_{-1}, x_0, x_1, \ldots\rangle / \langle x_{i+1}x_i - (-1)^i x_i x_{i+1},\; x_i x_j + x_j x_i\,;\; |i-j| > 1\rangle.$$

What Poisson structure, Hamiltonian derivations, if any, correspond to this limit? Can we present this hierarchy on $\mathcal{A}$ in the Hamiltonian form:

$$\frac{dx_\ell}{dt_{2m}} = \{\mathbf{H}_{2m}; x_\ell\}, \quad \{\mathbf{H}_{2m}, \mathbf{H}_{2n}\} = 0\,?$$

Definition of Poisson algebra

Definition

Let $\mathcal{A}$ be any (unitary) associative algebra over a commutative ring R. A skew-symmetric R-bilinear map $\{\cdot\,,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be a Poisson bracket on $\mathcal{A}$ when it satisfies the Jacobi and Leibniz identities: for all $a, b, c \in \mathcal{A}$,

$$\begin{array}{lll} (1) & \{\{a,b\}\,,c\} + \{\{b,c\}\,,a\} + \{\{c,a\}\,,b\} = 0\ , & \text{(Jacobi identity)},\\ (2) & \{a,bc\} = \{a,b\}\,c + b\,\{a,c\}\ , & \text{(Leibniz identity)}. \end{array}$$

$(\mathcal{A}, \{\cdot\,,\cdot\})$ or $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ is then said to be a Poisson algebra (over R). When $\mathcal{A}$ is commutative one says that the Poisson algebra $(\mathcal{A}, \{\cdot\,,\cdot\})$ is commutative.

Any associative algebra $\mathcal{A}$ has a natural Poisson bracket, given by the commutator $\{a, b\} := [a, b]$. Thus, $(\mathcal{A}, [\cdot, \cdot])$ is a Poisson algebra.

Some noncommutative history remarks

1998: Farkas and Letzter proved that for any prime Poisson algebra $\mathcal{A}$, which is not commutative, the Poisson bracket must be the commutator in $\mathcal{A}$, up to an appropriate scalar factor.

2004: Van den Bergh introduced double Poisson bracket $\{\{\cdot,\cdot\}\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying a modified skew-symmetry condition and modified Jacobi identity, and defined a double Poisson algebra.

2005: Crawley-Boevey studied non-commutative Poisson structures
2007: Crawley-Boevey, Etingof and Ginzburg introduced double derivations and Hamiltonian reductions

1998: Olver and Sokolov, 2000: Olver and Wang introduced and studied Hamiltonian structure of integrable PDEs on free associative algebras

2000: AVM and Sokolov introduced and studied Hamiltonian structure of ODEs on free associative algebras and "Poisson brackets" on $\mathcal{A}^{\natural} = \mathcal{A}/[\mathcal{A},\mathcal{A}]$.

2019: De Sole, Kac, Valeri and Wakimoto introduced Local and Non-local Multiplicative Poisson Vertex Algebras.

Kontsevich, Efimovskaya, Wolf, Chalykh, Fairon, Casati, Wang, ...

[Just appeared in arXiv, May 28, 2024]: Reshetikhin with Liashyk and Sechin.

Formal deformation $(\mathcal{A}[[\nu]], \star)$ of and associative algebra $\mathcal{A}$

Let $\mathcal{A}$ be any associative algebra over R.
$\mathcal{A}[[\nu]]$, the $R[[\nu]]$-module of formal power series in ν. Any element $A \in \mathcal{A}[[\nu]]$ can be written in a unique way as

$$A = a_0 + \nu a_1 + \nu^2 a_2 + \cdots , \qquad a_i \in \mathcal{A} .$$

Definition

Suppose that $\mathcal{A}[[\nu]]$ is equipped with the structure of an associative algebra over $R[[\nu]]$, with product denoted by $\star$. Then $(\mathcal{A}[[\nu]], \star)$, or simply $\mathcal{A}[[\nu]]$, is said to be a (formal) deformation of $\mathcal{A}$ if for any $a, b \in \mathcal{A}$,
$a \star b = ab + \mathcal{O}(\nu)$, i.e., $a \star b - ab \in \nu\mathcal{A}[[\nu]]$.
$\Leftrightarrow$
Under the natural identification of $\mathcal{A}$ with $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]]$ the canonical projection $\pi : (\mathcal{A}[[\nu]], \star) \to (\mathcal{A}, \cdot)$ is a morphism of algebras.
Here π can be seen as evaluation at $\nu = 0$.

The commutator in $\mathcal{A}[[\nu]]$ is denoted by $[A, B]_\star := A \star B - B \star A$.

The R-bilinear maps $(\cdot, \cdot)_i, \{\cdot\, , \cdot\}_i : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are defined by

$$a \star b = ab + \nu(a, b)_1 + \nu^2(a, b)_2 + \cdots , \qquad a, b \in \mathcal{A} \subset \mathcal{A}[[\nu]]$$
$$[a, b]_\star = [a, b] + \nu \{a, b\}_1 + \nu^2 \{a, b\}_2 + \cdots , \qquad \{a, b\}_i = (a, b)_i - (b, a)_i .$$

Poisson subalgebra, ideal and quotient algebra

- When $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a Poisson algebra and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ which is also a Lie subalgebra of $(\mathcal{A}, \{\cdot\,,\cdot\})$, then $(\mathcal{B}, \{\cdot\,,\cdot\})$ is a Poisson algebra; we say that $(\mathcal{B}, \{\cdot\,,\cdot\})$ is a Poisson subalgebra of $\mathcal{A}$.
- Similarly, if $\mathcal{I}$ is an ideal of $\mathcal{A}$ which is also a Lie ideal of $(\mathcal{A}, \{\cdot\,,\cdot\})$ then $\mathcal{I}$ is a Poisson ideal of $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ is a Poisson algebra; we say that it is a quotient Poisson algebra of $\mathcal{A}$.

Example: Let $\mathcal{A}[[\nu]]$ be a deformation of a commutative R-algebra $\mathcal{A}$.

$(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$ is the corresponding natural Poisson algebra.

$\Big[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\star \subset \nu\mathcal{A}[[\nu]]$. Thus $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\nu)$ is a Poisson algebra, where the bracket $[A,B]_\nu = \frac{1}{\nu}[A,B]_\star$ is well defined.

The ideal $\nu\mathcal{A}[[\nu]] \subset (\mathcal{A}[[\nu]], [\cdot,\cdot]_\nu)$ is also a Lie ideal:

$$\Big[\nu\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\nu = \Big[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\Big]_\star \subset \nu\mathcal{A}[[\nu]].$$

Thus $\mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]] = \mathcal{A}$ is a Poisson algebra with the Poisson bracket

$$\{a,b\} = \{a,b\}_1 \quad \left(= \lim_{\nu\to 0} \frac{a\star b - b\star a}{\nu}\right).$$

Algebra $\mathcal{A}$ which is not necessarily commutative, and $Z(\mathcal{A})$ is its centre.
$\mathcal{A}[[\nu]]$ is a deformation of $\mathcal{A}$ and $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$ is its natural Poisson algebra.
Since $\left[\mathcal{A}[[\nu]], \mathcal{A}[[\nu]]\right]_\star \subset \mathcal{A}[[\nu]]$, and not $\nu\mathcal{A}[[\nu]]$, we cannot introduce $[\cdot,\cdot]_\nu$.

We define $\mathcal{H}_\nu = Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]]$. Quantum Hamiltonians live in $\mathcal{H}_\nu$.

We proved the following statements:

- $\mathcal{H}_\nu$ is a Poisson subalgebra of $(\mathcal{A}[[\nu]], [\cdot,\cdot]_\star)$, i.e. $[\mathcal{H}_\nu, \mathcal{H}_\nu]_\star \subset \mathcal{H}_\nu$.
- $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$ is a Poisson algebra, i.e.
  $[\mathcal{H}_\nu, \mathcal{H}_\nu]_\star \subset \nu\mathcal{H}_\nu \Leftrightarrow \{Z(\mathcal{A}), Z(\mathcal{A})\}_1 \subset Z(\mathcal{A})$.
- $\nu^2\mathcal{A}[[\nu]]$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$.
- $\mathcal{H}_\nu/\nu^2\mathcal{A}[[\nu]] \simeq Z(\mathcal{A}) \times \mathcal{A}$ is a (noncommutative) Poisson algebra.
- $\nu\mathcal{H}_\nu$ is a Poisson ideal of $(\mathcal{H}_\nu, [\cdot,\cdot]_\nu)$.
- $\mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}$ is a commutative Poisson algebra.

Commutative Poisson algebra $\Pi(\mathcal{A}) = Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})}$

$$A \in \mathcal{H}_\nu = Z(\mathcal{A}) + \nu\mathcal{A}[[\nu]] \;\Rightarrow\; A = a_0 + \nu a_1 + \nu^2 a_2 + \cdots\,, \quad a_0 \in Z(\mathcal{A}),\; a_1, a_2, \ldots \in \mathcal{A}.$$

$$\pi_\Pi : \mathcal{H}_\nu / \nu\mathcal{H}_\nu \;\to\; \Pi(\mathcal{A}) := Z(\mathcal{A}) \times \frac{\mathcal{A}}{Z(\mathcal{A})} \;\; \Big(\text{Canonical projection}\Big).$$

$$A \in \Pi(\mathcal{A}) \;\Rightarrow\; A = (a_0, a_1 + Z(\mathcal{A})) = (a_0, \overline{a_1}), \quad a_0 \in Z(\mathcal{A}),\; a_1 \in \mathcal{A}$$

Proposition

$(\Pi(\mathcal{A}), \{\cdot\,,\cdot\})$ is a commutative Poisson algebra with associative multiplication $\cdot$ and Poisson bracket $\{\cdot\,,\cdot\}$:

$$(a, \overline{a_1}) \cdot (b, \overline{b_1}) = \Big(ab, \overline{ab_1 + a_1 b + (a,b)_1}\Big)\,,$$

$$\Big\{(a, \overline{a_1}), (b, \overline{b_1})\Big\} = \Big(\{a,b\}_1\,, \overline{\{a,b\}_2 + \{a_1,b\}_1 + \{a,b_1\}_1 + [a_1,b_1]}\Big)\,,$$

for all $(a, \overline{a_1}),\ (b, \overline{b_1}) \in \Pi(\mathcal{A})$.

When $\mathcal{A}$ is commutative, $Z(\mathcal{A}) = \mathcal{A}$ and $\mathcal{H}_\nu = \mathcal{A}[[\nu]]$

$$\Pi(\mathcal{A}) \simeq \mathcal{H}_\nu/\nu\mathcal{H}_\nu \simeq \mathcal{A}[[\nu]]/\nu\mathcal{A}[[\nu]] \simeq \mathcal{A}\,, \quad \{\cdot\,,\cdot\} = \{\cdot\,,\cdot\}_1\,.$$

Heisenberg and Hamiltonian derivations

The Heisenberg derivation $\delta_{\hat{H}} : \mathcal{A}[[\nu]] \to \mathcal{A}[[\nu]]$:

$$\delta_{\hat{H}}(a) := \frac{1}{\nu}[\hat{H}, a]_\star$$

with $\hat{H} = H_0 + \nu H_1 + \nu^2 H_2 + \cdots \in \mathcal{A}[[\nu]]$ is well defined (admits a finite limit, as $\nu \to 0$, for any $a \in \mathcal{A}$) if and only if $H_0 \in Z(\mathcal{A})$.

Thus $\hat{H} \in \mathcal{H}_\nu$, and

$$\lim_{\nu \to 0} \delta_{\hat{H}}(a) = \lim_{\nu \to 0} \frac{1}{\nu}[H_0 + \nu H_1 + \nu^2 H_2 + \cdots, a]_\star = \{H_0, a\}_1 + [H_1, a].$$

For $\mathbf{H} = (H_0, \overline{H_1}) \in \Pi(\mathcal{A})$ we define a Hamiltonian derivation $\partial_{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$

$$\partial_{\mathbf{H}}(a) = \{H_0, a\}_1 + [H_1, a].$$

We have shown that $\mathcal{A}$ is a Poisson module over $(\Pi(\mathcal{A}), \cdot, \{\cdot, \cdot\})$, with actions given for $(a, \overline{a_1}) \in \Pi(\mathcal{A})$ and $b \in \mathcal{A}$ by

$$(a, \overline{a_1}) \cdot b = b \cdot (a, \overline{a_1}) = ba = ab \,, \quad \{(a, \overline{a_1}); b\} = \{a, b\}_1 + [a_1, b] \,.$$

Definition

Let $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ be a Poisson algebra over R and let M be an R-module. Then M is said to be a $\mathcal{A}$-Poisson module (or Poisson module over $\mathcal{A}$ or over $(\mathcal{A}, \{\cdot\,,\cdot\})$) when M is both a $(\mathcal{A}, \cdot)$-bimodule and a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module, satisfying the following derivation properties: for all $a, b \in \mathcal{A}$ and $m \in M$,

$$\begin{aligned} \{a\,; b\cdot m\} &= \{a, b\}\cdot m + b\cdot\{a\,; m\}\ , \\ \{a\,; m\cdot b\} &= m\cdot\{a, b\} + \{a\,; m\}\cdot b\ , \\ \{a\cdot b\,; m\} &= a\cdot\{b\,; m\} + \{a\,; m\}\cdot b\ . \end{aligned}$$

In the above formulas, the three actions of $\mathcal{A}$ on M have been written $a\cdot m$, $m\cdot a$ and $\{a\,; m\}$ for $a \in \mathcal{A}$ and $m \in M$. In this notation, the fact that M is a $\mathcal{A}$-bimodule (respectively a $(\mathcal{A}, \{\cdot\,,\cdot\})$-Lie module), takes the form

$$a\cdot(b\cdot m) = (a\cdot b)\cdot m\ , \quad (m\cdot a)\cdot b = m\cdot(a\cdot b)\ , \quad a\cdot(m\cdot b) = (a\cdot m)\cdot b\ ,$$

$$\{\{a, b\}\,; m\} = \{a\,; \{b\,; m\}\} - \{b\,; \{a\,; m\}\}\ ,$$

for $a, b \in \mathcal{A}$ and $m \in M$.

Non-Abelian Hamiltonian Equations

In our setting, let a Hamiltonian $\mathbf{H} = (H_0, \overline{H_1}) \in \Pi(\mathcal{A})$ and $a \in \mathcal{A}$. Then the corresponding non-Abelian Hamiltonian equation on $\mathcal{A}$ is defined as

$$\frac{da}{dt} = \{\mathbf{H}\,; a\} = \{H_0, a\}_1 + [H_1, a].$$

Proposition

Suppose that $\mathbf{F} = (F_0, \overline{F_1}), \mathbf{G} = (G_0, \overline{G_1}) \in \Pi(\mathcal{A})$. Then

$$\partial_{\mathbf{F}}\partial_{\mathbf{G}} - \partial_{\mathbf{G}}\partial_{\mathbf{F}} = \partial_{\{\mathbf{F},\mathbf{G}\}}.$$

In particular, if $\mathbf{F}$ and $\mathbf{G}$ are in involution, $\{\mathbf{F}, \mathbf{G}\} = 0$, their associated derivations $\partial_{\mathbf{F}}$ and $\partial_{\mathbf{G}}$ of $\mathcal{A}$ commute.

We have two types of derivations:

$$\partial_{\mathbf{H}} = \{\mathbf{H}\,; \cdot\} \,:\, \mathcal{A} \to \mathcal{A}, \quad \text{and} \quad \partial'_{\mathbf{H}} = \{\mathbf{H}\,, \cdot\} \,:\, \Pi(\mathcal{A}) \to \Pi(\mathcal{A})$$

When $\mathcal{A}$ is not commutative, these derivations are defined on different algebras and none of the two determines the other one.

When $\mathcal{A}$ is commutative, $\partial_{\mathbf{H}}$ and $\partial'_{\mathbf{H}}$ are both derivations of $\mathcal{A}$ and $\partial_{\mathbf{H}} = \partial'_{\mathbf{H}}$.

The advantage of having a commutative Poisson algebra and module

In the case of a commutative Poisson algebra $(\Pi(\mathcal{A}), \{\cdot, \cdot\})$ generated by the set $X_1, \ldots, X_M$, it is sufficient to find the Poisson brackets $\{X_i, X_j\}$. Then for any $P, Q \in \Pi(\mathcal{A})$

$$\{P, Q\} = \sum_{i,j=1}^{M} \frac{\partial P}{\partial X_i} \frac{\partial Q}{\partial X_j} \{X_i, X_j\} \ .$$

To compute Hamiltonian derivations, it is sufficient to find the table $\partial_{X_i}(y_k) = \{X_i\,;\, y_k\}$, where $y_1, \ldots$ are generators of the algebra $\mathcal{A}$.

Example: quantum plane at $q = -1 + \nu$

Quantum plane:

$$\mathcal{A}_q = \frac{\mathbb{C}(q)\langle x, y\rangle}{\langle yx - q\, xy\rangle}.$$

If $q^N \neq 1$ for some $N \in \mathbb{N}$, then $Z(\mathcal{A}_q) = \mathbb{C}$.

Quantum plane at $q = -1 + \nu$

$$\frac{\mathbb{C}[[\nu]]\langle x, y\rangle}{\langle yx - (\nu - 1)xy\rangle} \simeq \mathcal{A}[[\nu]] \text{ , where } \mathcal{A} := \frac{\mathbb{C}\langle x, y\rangle}{\langle yx + xy\rangle} \text{ .}$$

The product $\star$ on $\mathcal{A}[[\nu]]$ is defined by

$$y{\star}x = (\nu{-}1)x{\star}y = (\nu{-}1)xy, \quad (x, y)_1 = 0,\ (y, x)_1 = xy,\ (x, y)_k = (y, x)_k = 0,\ k \geqslant 2,$$

and associativity.

$\mathcal{A}$ is generated by x, y, satisfying the condition $yx = -xy$.
$Z(\mathcal{A})$ is generated by x^2, y^2.
$\mathcal{A}/Z(\mathcal{A})$ is generated as a $Z(\mathcal{A})$-module by $\overline{x}$, $\overline{y}$ and $\overline{xy}$.
$\Pi(\mathcal{A})$ is generated by 5 elements:

$$X = \left(x^2, \overline{0}\right) \text{ , } Y = \left(y^2, \overline{0}\right) \text{ , } U = (0, \overline{x}) \text{ , } V = (0, \overline{y}) \text{ , } W = (0, \overline{xy}) \text{ ,}$$

Example: quantum plane at $q = -1 + \nu$

$$X = \left(x^2, \overline{0}\right) \;,\; Y = \left(y^2, \overline{0}\right) \;,\; U = (0, \overline{x}) \;,\; V = (0, \overline{y}) \;,\; W = (0, \overline{xy}) \;,$$

$\Pi(\mathcal{A}) \simeq \mathbb{C}[X, Y, U, V, W]/\langle U^2, V^2, W^2, UV, VW, UW \rangle$.

Poisson brackets between the generators of $\Pi(\mathcal{A})$:

$\{\cdot,\cdot\}$	X	Y	U	V	W
X	0	$4X \cdot Y$	0	$2X \cdot V$	$2X \cdot W$
Y	$-4X \cdot Y$	0	$-2U \cdot Y$	0	$-2Y \cdot W$
U	0	$2U \cdot Y$	0	$2W$	$2X \cdot V$
V	$-2X \cdot V$	0	$-2W$	0	$-2Y \cdot U$
W	$-2X \cdot W$	$2Y \cdot W$	$-2X \cdot V$	$2Y \cdot U$	0

Example: quantum plane at $q = -1 + \nu$

In the $\Pi(\mathcal{A})$-Poisson module $\mathcal{A}$ we have:

$\cdot$	x	y
X	x^3	x^2y
Y	xy^2	y^3
U	0	0
V	0	0
W	0	0

$\{\cdot;\cdot\}$	x	y
X	0	$2x^2y$
Y	$-2xy^2$	0
U	0	$2xy$
V	$-2xy$	0
W	$-2x^2y$	$2xy^2$

$$\hat{H} = -\frac{1}{2-\nu}(y^2 - \nu(-1+\nu))xy + (-1+\nu)^2x^2)$$

$$\mathbf{H} = -\frac{1}{2}(x^2 + y^2, \overline{xy}) = -\frac{1}{2}(X + Y + W), \quad \frac{dx}{dt} = \{\mathbf{H}; x\} = xy^2 + x^2y.$$

What is integrability of non-Abelian Hamiltonian systems?

In the commutative classical world, a Hamiltonian system in $\mathbb{R}^N$ is Liouville integrable if it has n functionally independent first integrals $H_1, \ldots H_n$ in involution $\{H_p, H_k\} = 0,\ 1 \leqslant p, k \leqslant n$ and $N - 2n$ functionally independent Casimir elements of the Poisson bracket.

In the corresponding quantum system, the integrability is often identified with the existence of n commuting and algebraically independent elements of the quantum algebra $\mathcal{A}_\hbar$ and $N - 2n$ generators of its center $Z(\mathcal{A}_\hbar)$.

It is evident that the central elements of $\mathcal{A}[[\nu]]$ give rise to Casimir elements within the Poisson algebra $\Pi(\mathcal{A})$. It would be logical to propose a definition of integrability for a non-Abelian Hamiltonian system, requiring the existence of n algebraically independent elements $\mathbf{H}_k \in \Pi(\mathcal{A})$ in involution $\{\mathbf{H}_p\,, \mathbf{H}_k\} = 0,\ 1 \leqslant p < k \leqslant n$, along with $N - 2n$ independent Casimir elements within the Poisson algebra $\Pi(\mathcal{A})$.

The problem of solutions for non-Abelian Hamiltonian systems is wide open.

A Poem About Pure Math

A Poem about Pure Math

Vladimir Zakharov

He did not waste his time in seedy pubs,
Did Dr Hardy, the pure mathematician.
On the green lawns of Cambridge
He strolled, together with Ramanujan,
More often alone, always thinking of numbers,
Prime and perfect numbers.

The First World War, then the Second,
Epochs rose and fell,
But prime numbers are still prime,
And the perfect ones, my friend,
Have lost none of their marvellous perfection.

There is something everlasting in this world!

Translated by Dr. B. Phibbs
Amended by R. Davies

1996

Не слонялся по притонам злачным

Доктор Харди, чистый математик.

В Кембридже зеленом по лужайкам

Он гулял — вдвоем с Рамануджаном,

Больше же один. И все о числах

Думал он, простых и совершенных.

Первая, Вторая мировая,

Поднялись и рухнули эпохи.

Но простые числа так же просты,

И от совершенных не убыло

Дивного, мой друг, их совершенства.

Есть же нечто прочное на свете!